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A random walk approach to the Bak–Sneppen evolution model

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Abstract. The dynamics of the avalanche width in the recently proposed evolution model is described using a random walk picture. In this approach the critical exponents for avalanche distribution and avalanche average time are found to be the same as in the previous mean-field approximation whereas the critical value of the fitness is in perfect agreement to previous numerical estimates. A continuous time random walk picture is studied as a possible way to improve the mean-field treatment.

0. Introduction

In Nature there are non-equilibrium systems which present a scale invariant distribution for the events and the structures produced during their evolution. They can be found in fields such as interface growth, astrophysics, geophysics, biological evolution, the stock market, etc [1, 2, 6]. In the last decade stochastic models were proposed which are able to exhibit this kind of behaviour starting from simple dynamical rules between elementary agents. The dynamics of these models is extremal; such a system evolves following the global extremal value of some relevant parameter [1, 6].

The Bak–Sneppen (BS) model of evolution has been extracted from more a complex model which describes the evolution of genotype in the fitness landscape [8]. The model is an alternative to the catastrophic theory of evolution since it can produce large extinction events in the ecosystem using its own rules, while catastrophic theory claims that external factors are the first cause for large extinction events recorded during Earth history. The mathematical simplicity of the model has attracted numerical studies [2, 5, 7] and mean-field analytic treatments [3, 4].

The model treats a number of N species interacting on a one-dimensional chain (a simple picture of the food chain). Each species is assigned a scalar parameter called the fitness with values in the interval $(0, 1)$. It is thought of as a measure for the adaptability of a species to the ecosystem in a coarse grain description. One step of the dynamics consists of choosing the site with the smallest fitness, then new random independent values from the $(0, 1)$ interval are attributed to this site and to its two neighbours with a uniform distribution. We define a λ-avalanche $(0 < \lambda < 1)$ as the number of steps between two consecutive configurations with all the fitness values greater than λ. Numerically it has been found that

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the stationary state exhibits scaling laws in the $N \to \infty$ limit for the avalanches temporal distribution at $\lambda_{\text{critical}} \approx 0.667$ [5]

$$p(t) \approx t^{-r} \qquad \tau \to \infty$$

with $\tau \approx 1.07$. The mean avalanche temporal size diverges as $\lambda \to \lambda_{\text{critical}}$

$$\bar{t} \approx |\lambda - \lambda_{\text{critical}}|^{-\gamma}$$

with $\gamma \approx 2.7$ [5]. The model also shows critical behaviour in the versions with more than one dimension or with slightly modified dynamics [5, 7].

In this paper we propose a new approximated solution. Our treatment focuses on the dynamics of the avalanche spatial width in a mean-field approach. In the BS model the avalanche width is a random variable having memory effects; this is removed by updating all the sites within an avalanche (see section 2 for more details), and thus we keep track only of the spatial extension of the avalanches. This is not possible in the infinite range approximation [4]. We found the value of $\lambda_{\text{critical}} = \frac{2}{3}$, which is very close to the numerical result, whereas in the infinite range model $\lambda_{\text{critical}} = \frac{1}{3}$.

This paper is organized as follows. Section 1 presents the general master equation for the fitness distribution and the derivation of the mean-field equation found with probabilistic arguments in [2]. Section 2 introduces our analytical treatment based on a mapping in a random walk problem. In this approximation we compute exactly the value of $\lambda_{\text{critical}}$ and the critical exponents τ and γ as they were defined in [5]. Section 3 presents a way to improve the method introducing a continuous time random walk description. A numerical study shows the improvement of the critical exponent τ but $\lambda_{\text{critical}}$ remains fixed. The details of the calculation are given in the appendix.

1. The master equation

The BS model is completely characterized by the probability $P(x_1, x_2, \ldots, x_N; t)$ of finding the system in the state $(x_1, x_2, \ldots, x_N)$ at the time t given the initial distribution at $t = 0$. Because there are no memory effects, the evolution of the system is described by the following master equation,

$$P(x_1, x_2, \ldots, x_N; t+1) = \sum_i \int \mathrm{d}x'_i \, \mathrm{d}x'_{i-1} \, \mathrm{d}x'_{i+1} P_{\text{st}} P(i; x_1, \ldots, x'_{i-1}, x'_i, x'_{i+1}, \ldots, x_N) \times P(x_1, \ldots, x'_{i-1}, x'_i, x'_{i+1}, \ldots, x_N; t) \qquad (1)$$

where periodic boundary conditions were assumed and $P_{\text{st}}(i; x_1, \ldots, x'_{i-1}, x'_i, x'_{i+1}, \ldots, x_N)$ is the probability to have activity at site i if the system is in the configuration $(x_1, \ldots, x'_{i-1}, x'_i, x'_{i+1}, \ldots, x_N)$. For the original one-dimensional BS model

$$P_{\text{st}}(i; x_1, x_2, \ldots, x_N) = \prod_{j \neq i} \theta(x_j - x_i) \qquad (2)$$

where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } \leqslant 0. \end{cases}$$

At stationarity, integrating in (1) over $x_2, \ldots, x_N$ we easily obtain the following relation,

$$P_{\text{ac}}(1, x_1) + P_{\text{ac}}(2, x_1) + P_{\text{ac}}(N, x_1) - \frac{3}{N} = 0 \qquad (3)$$

where

$$P_{\mathrm{ac}}(i; xj) = \int \mathrm{d}x_1 \ldots \mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1} \ldots \mathrm{d}x_N\, P_{\mathrm{st}}(i; x_1, \ldots, x_N) P(x_1, \ldots, x_N)$$

is the probability of having activity in site i when in site j the fitness has the value x_j. If in equation (3) we try a stationary self-consistent mean-field solution of the form $p(x_1, \ldots, x_N) = p(x_1)p(x_2) \ldots p(x_N)$, after some algebra we obtain

$$\left(1 - \frac{2}{N-1}\right) Q^N(x) + \frac{2N}{N-1} Q(x) + 3x - 3 = 0 \qquad (4)$$

with $Q(x) = \int_x^1 p(x')\,\mathrm{d}x'$. Equation (4) was previously obtained in [3], in the $N \to \infty$ limit one finds $p(x) = \frac{3}{2}, x \in (\lambda_{\mathrm{critical}}, 1)$ and $p(x) = 0$ when $x \in (0, \lambda_{\mathrm{critical}}), \lambda_{\mathrm{critical}} = \frac{1}{3}$ whereas numerically $\lambda_{\mathrm{critical}} \approx \frac{2}{3}$ [2]. The statistical independence between the sites in the mean-field solution allows the reduction of the problem at a one-dimensional random walk on the positive semi-axis where the state n represents the state of the system with n fitness values greater then λ. The solution developed in [4] gives the same $\lambda_{\mathrm{critical}}$ as predicted by equation (4) and the critical exponents $\tau = \frac{3}{2}, \lambda = 1$.

2. The new approach

The size of a λ-avalanche is defined as the number of steps between two consecutive events with no fitness below the value λ; hence, it is a quantity characterizing time intervals. For a system of size N, with free boundary conditions, we define the avalanche width at a given time t as the number of sites between the left most species with the fitness lees than λ and the right most species with the fitness less than λ. The species between these two sites can have any value of the fitness. This is a quantity which characterizes the spatial structure of the system. The width evolution has memory effects for the originally proposed dynamics; in the spirit of the mean field we approximate the evolution of the avalanche width as follows: at every step the species between the right and left extremities of the avalanche are updated independently and we also update the nearest neighbour site at the right or left extremity of the avalanche accounting for the fact that in the original dynamics an avalanche can increase its width only with one site. The complete randomness makes the movements of the two extrema completely equivalent and for this reason we choose to move only in one direction. At the origin we also accept a two sites step.

In this approach the avalanche width is a random variable without memory effects on a discreet set of states which now can be extended to the entire non-negative semi-axis with the zero state corresponding to the state with no species blow λ and the state n to a realization of the BS model with n sites between the left most and the right most sites with the value of their fitness less than λ. The evolution rules produce the following transition matrix for the avalanche width:

$$p = \begin{pmatrix} (1-\lambda)^2 & 2\lambda(1-\lambda) & \lambda^2 & 0 & 0 & \ldots \\ (1-\lambda)^2 & 2\lambda(1-\lambda) & \lambda^2 & 0 & 0 & \ldots \\ (1-\lambda)^3 & 3\lambda(1-\lambda)^2 & 2\lambda^2(1-\lambda) & \lambda^2 & 0 & \ldots \\ (1-\lambda)^4 & 4\lambda(1-\lambda)^3 & 3\lambda^2(1-\lambda)^2 & 2\lambda^2(1-\lambda) & \lambda^2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \qquad (5)$$

The formulae for the matrix elements are

$$p_{00} = (1-\lambda)^2 \qquad p_{01} = 2\lambda(1-\lambda) \qquad p_{02} = \lambda^2 \qquad p_{0j} = 0 \qquad j > 2$$

$$p_{j0} = (1-\lambda)^{j+1} \qquad j \geqslant 1$$

$$p_{j1} = (j+1)\lambda(1-\lambda)^j \qquad j \geqslant 1$$

$$p_{j2} = j\lambda^2(1-\lambda)^{j-1} \qquad j \geqslant 1$$

$$p_{jl} = \begin{cases} p_{j+1,i-1} & \text{if } l \geqslant 2 \text{ and } j \leqslant l-1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

with the convention that p_{ij} is the transition probability from the state i to the state j and $i, j \in \{0, 1, 2, \ldots\}$. The distribution probability of avalanches is the first return probability distribution for the above-defined random walk and it can be written as

$$p(n+2) = \sum_{i=1}^{\infty} p_{01}\tilde{p}_{1i}^{(n)} p_{i0} + \sum_{i=1}^{\infty} p_{02}\tilde{p}_{2i}^{(n)} p_{i0} \tag{7}$$

where $\tilde{p}_{1i}^{(n)}$ of the i, j element of the nth power of the matrix $\tilde{p}$ describing the evolution of the random walk outside of the origin. The matrix $\tilde{p}_{ij}^{(n)}$ is obtained from the matrix $\tilde{p}$ by setting to zero its first row and first column. The first (second) term on the right-hand side of equation (7) represents the first return probability after n steps when the initial step is single (double).

Since we are concerned only with the asymptotic behaviour of the model we can modify the first two columns of the transition matrix p so as to have the same elements on the diagonals of the $\tilde{p}$ matrix. Keeping the closure relation $\sum_j \tilde{p}_{ij} = 1$ we produce the following matrix:

$$p' = \begin{pmatrix} (1-\lambda)^2 & 2\lambda(1-\lambda) & \lambda^2 & 0 & 0 & \ldots \\ (1-\lambda)^2(1+2\lambda) & 2\lambda^2(1-\lambda) & \lambda^2 & 0 & 0 & \ldots \\ (1-\lambda)^3(1+3\lambda) & 3\lambda^2(1-\lambda)^2 & 2\lambda^2(1-\lambda) & \lambda^2 & 0 & \ldots \\ (1-\lambda)^4(1+4\lambda) & 4\lambda^2(1-\lambda)^3 & 3\lambda^2(1-\lambda)^2 & 2\lambda^2(1-\lambda) & \lambda^2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{8}$$

The asymptotic behaviour of the first return time distribution is the same for both random walks described by the matrices p and p'. In fact, in equation (7) we can go a step further developing $\tilde{p}_{1i}^{(n)}$ with respect to site 1. The remaining matrix from the second column and second row is identical with the $\tilde{p}'$ matrix obtained from p' in the same way as $\tilde{p}$ from p:

$$\tilde{p}_{1i}^{(n)} = \sum_{\substack{n_1+\cdots+n_j \\ =n-n'}} \prod_{l=1}^{j} \left(\sum_{k=2}^{\infty} \tilde{p}_{12}\tilde{p}_{2k}^{(n_l)}\tilde{p}_{k1} \right) ((1-\delta_{1i})\tilde{p}_{12}\tilde{p}_{2i}^{(n')} + \delta_{1i}\tilde{p}_{11}^{(n')}) \tag{9}$$

with $n = n' + n_1 + \cdots + n_j$ and δ_{ij} the Kronecker symbol. The terms in equation (9) represent multiple returns on site 1 before the last step to the origin. In the $n \to \infty$ limit the terms with n' and all the n_j bounded but one have the same asymptotic behaviour as $\tilde{p}_{1i}^{(n)}$ because they are generated by the same matrix; the other terms will decay exponentially, due to the $\tilde{p}_{12}^{j}$ factor, or as a power of the leading term when there are two or more unbounded indices. If $i = 1, n'$ has to be bounded to avoid the exponential decay.

In the appendix we present the computation for the generating function of the avalanche distribution probability $R(z) = \sum_t z^t P(t)$ (A15). The average time for the avalanche distribution is

$$\bar{t} = \left.\frac{\mathrm{d}R(\xi)}{\mathrm{d}z}\right|_{z=1} = \left.\frac{\mathrm{d}R(\xi)}{\mathrm{d}(\xi)}\frac{\mathrm{d}\xi}{\mathrm{d}z}\right|_{z=1}. \tag{10}$$

Using equation (A15) we obtain that the mean time of the avalanche distribution can be written as

$$\bar{t} \approx |\lambda - \lambda_{\text{critical}}|^{-\gamma} \tag{11}$$

when $|\lambda - \lambda_{\text{critical}}| \ll 1$, with the critical exponent $\gamma = 1$ and the critical value of λ, $\lambda_{\text{critical}} = \frac{2}{3}$. We can also compute the asymptotic behaviour of the avalanches probability distribution using equation (A16) [10]. For $\lambda = \frac{2}{3}$ we found

$$p(t) \approx t^{-\gamma} \qquad t \to \infty \tag{12}$$

with $\tau = \frac{3}{2}$. For $\lambda \neq \frac{2}{3}$ the decay is exponential.

In the previous equations the critical exponent $\tau = \frac{3}{2}$ and γ have values as obtained in the mean-field solution [4], whereas $\lambda_{\text{critical}} = \frac{2}{3}$ is in extremely good agreement with the critical value of λ found in numerical simulations [2, 6, 7]. In the language of the Markov chain one can say that $\lambda = \frac{2}{3}$ is the transition point between persistent states ($\lambda \leqslant \frac{2}{3}$) and transient states ($\lambda > \frac{2}{3}$) [9]. Nevertheless λ is not a dynamical parameter for the BS model, since it introduces an 'observational window' for a certain variable which we may choose from the set of statistical variables compatible with the dynamics of the BS model. SOC appears when there is at least one statistical variable with events at all scale lengths. In our approach λ-avalanches are bounded to the origin for $\lambda < \frac{2}{3}$ and they escape to ∞ for $\lambda > \frac{2}{3}$. At $\lambda = \frac{2}{3}$ we have the peculiar stationary state in which the average time of avalanches is diverging; therefore, there are events on the all time scales.

3. Improving mean field

A significant difference between the random walk proposed in the previous section and the BS model consists of the fact that in the latter the system will spend a characteristic number of steps in a given state because the activity can appear between the leftmost and the rightmost sites where the fitness is less than λ. One possible way to improve our approximation is to promote the previous random walk to a continuous time random walk with inhomogeneous waiting time distributions, each waiting time distribution allowing for the persistence of a given size avalanche. The general equation for such a process can be written [10]

$$P_{ik}(t) = \delta_{ik}\,\mathrm{e}^{-c_i t} + \sum_{j=0}^{\infty} \int_0^t c_i\,\mathrm{e}^{-c_i t}\,p_{ij}\,P_{jk}(t - t')\,\mathrm{d}t' \tag{13}$$

where $P_{ik}(t)$ is the probability density of having the walker in state k at epoch t if at $t = 0$ it was in state i, p_{ij} are the elements of the p' matrix (8) and c_i^{-1} is the characteristic waiting time in site i and it represents the average lifetime for an avalanche of size i in the BS model. Intuitivity the average time of an avalanche is a function of the average number of sites with fitness less than λ which is increasing with the avalanche width; at criticality we propose the behaviour $c_i^{-1} \approx i^{\chi}$ for $i > 0$ and $c_0^{-1} = \alpha$ with α a given constant.

An avalanche is now defined as an off-time interval from the origin, whose probability distribution is independent of α [10]. The avalanche distribution function can be expressed as

$$p_{\text{av}}(t) = \sum_{n=1}^{\infty} p(n)\,p_n(t) \tag{14}$$

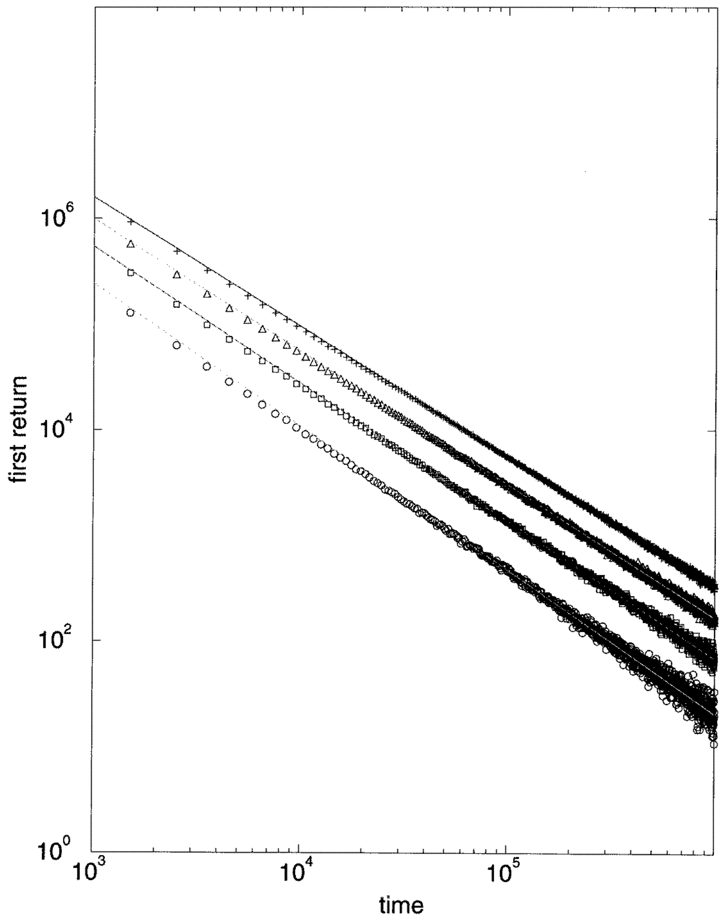

Figure 1. The first return distribution for $\chi = 1(\bigcirc)$, $\chi = 1.5(\square)$, $\chi = 2(\triangle)$, $\chi = 2.5(+)$. The obtained values for τ (table 1) are very well fitted by the formula $\tau = 1 + 1/(\chi + 2)$.

where $p(n)$ is the probability of an n step excursion out of the origin and it is the first return probability distribution for the random walk defined in section 2; $p_n(t)\,\mathrm{d}t$ is the probability of the first return to the origin in the interval $t, t + \mathrm{d}t$ after n steps. Intuitively we may say that for $\lambda < \frac{2}{3}$ the exponential decay of $p(n)$ will prohibit the long time avalanches and the average off-time will be finite [10]. If $\lambda = \frac{2}{3}$ there is a qualitative change; even if $p_n(t)$ decays exponentially the scale invariance of $p(n)$ for large n leads to critical behaviour.

We have performed a numerical simulation for the continuous time random walk for both transition matrices p and p' and we have found the same asymptotic behaviour at $\lambda_{\mathrm{critical}} = \frac{2}{3}$. The size of the lattice was large enough to avoid any size effects and the waiting time distribution functions have been chosen exponentials with a site-dependent mean lifetime $t_i = c_i^{-1} = i^{\chi}$. The simulations were made choosing four values $\chi = 1$, 1.5, 2, 2.5; the asymptotic behaviour is shown in figure 1. The numerical values for the critical exponent τ (table 1, figure 1) decreases monotonically as χ increases; a very good fit is given by the formula $\tau = 1 + 1/(\chi + 2)$. This behaviour is intuitively clear: the avalanche tend to last longer if the characteristic lifetimes grow faster and $\tau \to 1$ if $\chi \to \infty$.

Table 1. The numerical values of the critical exponent τ for four values of χ.

χ	τ
1.0	1.35 ± 0.01
1.5	1.30 ± 0.01
2.0	1.25 ± 0.01
2.5	1.22 ± 0.01

4. Conclusions

We have proposed a new approach to the Bak–Sneppen evolution model based on the dynamics of the avalanche width. The critical exponents are equal to those found previously in the mean-field solution [4], in fact they are universal properties of the one-dimensional random walk. The value $\lambda_{\text{critical}} = \frac{2}{3}$ is very close to the numerical results reported in [3, 7]. Therefore, we believe that the dynamics of the avalanche width contains useful information on the critical behaviour for this model. The structure of the generating function (A15) is intimately connected with the critical behaviour; the branch line appearing in the $N \to \infty$ limit generates the algebraic decay for the probability distribution of the avalanches. The continuous time random walk picture allows for a more careful analysis of the avalanche structure and it improves the τ critical exponent keeping $\lambda_{\text{critical}}$ at the same value; it also gives an intuitive decomposition of the algebraic decay distribution of the avalanches in a convolution of Poisson distributed events. The approximation can be extended to arbitrary dimensions. The active sites can be included in a minimal convex volume V. We update independently all the sites in V and the sites next to V. The maximum diameter of this set will have the dynamics described by the transition matrix p; therefore, the critical behaviour will remain unchanged.

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Appendix

We present the detailed calculation for the generating function of the avalanche probability distribution. We use the special form of the matrix $\bar{p}'$ obtained from the matrix p' (8) removing the line and the row with index zero. This matrix has equal diagonal elements and we can write it as a linear combination of one-diagonal matrices I_i defined as follows:

$$(I_i)_{kl} = \begin{cases} \delta_{k+i,l} & i \geqslant 0 \\ \delta_{k,l+i} & 1 < 0 \end{cases} \tag{A1}$$

I_0 being the identity matrix. From the definitions (A1) we can compute the commutator $T^{(i)} = I_1 I_{-i} - I_{-i} I_1$; for $i > 0$ we have

$$(T^{(i)})_{kl} = \delta_{i+1,1} \tag{A2}$$

and for $ij > 0$ we have the property

$$I_i I_j = I_{i+j}. \tag{A3}$$

The matrix $\bar{p}'$ can be expressed as

$$\bar{p}' = \lambda^2 I_1 + 2\lambda^2(1-\lambda)I_0 + \sum_{i=1}^{\infty}(i+2)\lambda^2(1-\lambda)^{i+1}I_{-i}$$
$$= \lambda^2 I_1 \sum_{i=0}^{\infty}(i+1)(1-\lambda)^i I_{-i} = \lambda^2 I_1 A \tag{A4}$$

where $A = \sum_{i=0}^{\infty}(i+1)(1-\lambda)^i I_{-i}$. Using equation (A3) it is easy to compute the nth power of this matrix

$$A^n = \sum_{j=0}^{\infty}\binom{2n+j-1}{j}(1-\lambda)^j I_{-j}. \tag{A5}$$

From equation (A2) we can compute the commutator $T_n = I_1 A_n - A_n I_1$ which has only the first column non-zero

$$(T_n)_{kl} = \binom{2n+k-1}{k}(1-\lambda)^k \delta_{l1}. \tag{A6}$$

Consequently,

$$(I_n T_n)_{j1} = \binom{3n+j-1}{j+n}(1-\lambda)^{j+n}. \tag{A7}$$

All the previous-mentioned properties lead us to the relation

$$(I_1 A)^n = I_n A^n - \sum_{i=0}^{n-2} I_i T_i (I_i A)^{n-i-2}. \tag{A8}$$

Equation (A8) implies the following equation for the generating matrix $G(z) = \sum_{i-0}^{\infty}(\lambda^2 I_1 A)^1 z^i$,

$$G(z) = F(z) - \sum_{i=1}^{\infty} I_i T_i \lambda^{2(i+1)} z^{i+1} G(z) \tag{A9}$$

where $F(z) = \sum_{i=1}^{\infty} I_i A^i z^i$, z a complex number. The sums which are appearing in equation (A9) can be performed in the following way:

$$u_j(z) = \sum_{k=1}^{\infty}(I_k T_k)_{j1}\lambda^{2(k+1)}z^{k+1} = \sum_{k=1}^{\infty}\binom{3k+j-1}{j+k}(1-\lambda)^{j+k}\lambda^{2(k+1)}z^{k+1}$$
$$= (1-\lambda)^{j-1}\xi^3 \sum_{k=1}^{\infty}\frac{1}{(k+j)!}(3k+j-1)\ldots 2k\xi^{2k-1}$$
$$= (1-\lambda)^{j-1}\xi^3 \sum_{k=1}^{\infty}\frac{1}{(k+j)!}\frac{\mathrm{d}^{k+j}}{\mathrm{d}\xi^{k+j}}\xi^{3k+j-1}$$
$$= (1-\lambda)^{j-1}\xi^3 \sum_{k=1}^{\infty}\frac{1}{2\pi \mathrm{i}}\oint_\Gamma \mathrm{d}\eta \frac{\eta^{3k+j-1}}{(\eta-\xi)^{j+k+1}}$$
$$= (1-\lambda)^{j-1}\xi^3 \frac{1}{2\pi \mathrm{i}}\oint_\Gamma \mathrm{d}\eta \frac{\eta^{j+2}}{(\eta-\xi)^{j+1}(-\eta^3+\eta-\xi)} \tag{A10}$$

where $\xi^2 = (1-\lambda)\lambda^2 z$. We can perform the summation if $|\eta^3/(\eta-\xi)| < 1$; this set is not empty for $0 < \xi < \frac{2}{3}\sqrt{\frac{1}{3}}$. There is an annulus with inner radius and external radius obtained from the positive solutions of the equation $(r+\xi)^3 - r = 0$; more over, one of the roots of the polynomial $-\eta^3 + \eta - \xi$ is inside the minimal integration contour Γ for $0 < \xi < \frac{2}{3}\sqrt{\frac{1}{3}}$ and the other two are outside the maximal integration contour. Using the above-mentioned properties of the matrices $\{I_k\}$ (A3) we can compute the elements of the matrix $I_n A^n$ which appear in the expression of the generating matrix $F(z)$:

$$(I_n A^n)_{1j} = \binom{3n-j}{n-j+1}(1-\lambda)^{n-j+1} \qquad (I_n A^n)_{21} = \binom{3n}{n+1}(1-\lambda)^{n+1}$$
$$(I_n A^n)_{2j} = (I_n A^n)_{1j-1} \qquad j > 1.$$

Equation (7) shows that we need to compute only the first two rows in the generating matrices $G(z)$ and $F(z)$. The general formula for these matrix elements of $F(z)$ is

$$F_{1j}(z) = \delta_{j1} + \sum_{n=j-1}^{\infty}(1-\lambda)^{n-j+1}\binom{3n-j}{n-j+1}\lambda^{2n}z^n.$$

The previous series can be summed following the same computational path as in equation (A10). If $j = 1$ we obtain

$$F_{1,1}(\xi) = 1 + \frac{\xi}{2\pi \mathrm{i}}\oint_\Gamma \frac{\eta^2}{(\eta-\xi)(-\eta^3+\eta-\xi)}.$$

For $j > 1$ we have the following expression:

$$F_{1j}(\xi) = \frac{\xi}{(1-\lambda)^{j-1}}\frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{\eta^{2j}}{-\eta^3+\eta-\xi}.$$

$F_{2j}(z) = F_{1j-1}(z)$, $j > 1$, because $I_n A^n$ has equal elements on diagonals, and by direct calculation

$$F_{21}(z) = \sum_{n=1}^{\infty}\binom{3n}{n+1}(1-\lambda)^{n+1}z^n = \frac{1-\lambda}{2\pi i}\xi\oint_\Gamma \frac{\eta^3}{(\eta-\xi)^2(-\eta^3+\eta-\xi)}.$$

In all the above formulae the contour Γ is the same as that used in equation (A10) and $\xi^2 = (1-\lambda)\lambda^2 z$. Solving equation (A9), we obtain for the first two rows the solutions in terms of the previously computed functions $u_1(z)$, $u_2(z)$, $F_{1j}(z)$ and F_{2j}:

$$G_{1,j}(z) = \frac{F_{1j}(z)}{1+u_1(z)}$$
$$G_{2,j}(z) = F_{2j}(z) - \frac{u_2(z)}{1+u_1(z)}F_{1j}(z).$$

The residue theorem allows us to compute the generating functions in terms of the third solutions of the polynomial $-\eta^3 + \eta - \xi$, $\eta_3(\xi)$, that solution which lies inside of the integration contour Γ in the above integrals:

$$G_{1,j}(\xi) = \frac{1}{(1-\lambda)^{j-1}}\frac{\eta_3(\xi)^{2j}}{\xi^2} \qquad i \geqslant 1 \tag{A11}$$

$$G_{2,j}(\xi) = -\frac{1}{(1+\lambda)^{j-2}}\left(\frac{1}{\xi^2}-2\right)\frac{\eta_3(\xi)^{2j}}{\xi^2} \qquad j \geqslant 1 \tag{A12}$$

with $\xi^2 = \lambda^2(1-\lambda)z$ and

$$\eta_3(\xi) = -\frac{1-\mathrm{i}\sqrt{3}}{2^{2/3}(-27\xi+\sqrt{729\xi^2-108})^{1/3}} - \frac{(1+\mathrm{i}\sqrt{3}(-27\xi+\sqrt{729\xi^2-108})^{1/3}}{62^{1/3}}. \tag{A13}$$

From equation (7) for the avalanche probability distribution one can write the generating function:

$$R(z) = (1-\lambda)^2 z + z^2 p_{01} \sum_{i=1}^{\infty} G_{1i}(z) p_{i0} + z^2 p_{02} \sum_{i=1}^{\infty} G_{2i}(z) p_{i0}$$
$$= (1-\lambda)^2 z + 2\lambda(1-\lambda) z^2 \sum_{i=1}^{\infty} \frac{1}{1+u_1(z)} F_{1i}(z)(1-\lambda)^{i+1}(1+(i+1)\lambda)$$
$$+ z^2\lambda^2 \sum_{i=1}^{\infty} \left(F_{2i}(z) - \frac{u_2(z)}{1+u_1(z)} F_{1i}(z) \right) (1-\lambda)^{i+1}(1+(i+1)\lambda). \qquad \text{(A14)}$$

The series which are appearing above can be summed and the closed expression for the generating function reads

$$R(\xi(z)) = -2\frac{1-\lambda}{\lambda}\xi^2 + \frac{2(1-\lambda)}{\lambda^3}\xi^2 \eta_3(\xi)^2 \left(1+2\lambda + \frac{\eta_3(\xi)^2}{1-\eta_3(\xi)^2}\left(1+\lambda\frac{3-2\eta_3(\xi)^2}{1-\eta_3(\xi)^2}\right)\right)$$
$$+ \frac{1-\lambda}{\lambda^2}(1-2\xi^2)\eta_3(\xi)^2 \left(1+2\lambda+(1+3\lambda)\eta_3(\xi)^2\right.$$
$$\left. + \frac{\eta_3(\xi)^4}{1-\eta_3(\xi)^2}\left(1+\lambda\frac{4-3\eta_3(\xi)^2}{1-\eta_3(\xi)^2}\right)\right). \qquad \text{(A15)}$$

Making the substitution $z = \mathrm{e}^{-s}$ we obtain

$$1 - R(s) \approx s^{1/2} \qquad \text{if } \lambda = \tfrac{2}{3}, s \to 0. \qquad \text{(A16)}$$

If $\lambda \neq \frac{2}{3}$, $R(s)$ is an analytical function at the origin.

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